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On $L(d, 1)$-labeling of Cartesian product of a cycle and a path

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Abstract

A k-$L(d, 1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to $\{0, 1, \dots, k\}$ such that $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$ and $|f(u) - f(v)| \geq d$ if $d(u, v) = 1$. The $L(d, 1)$-labeling problem is to find the $L(d, 1)$-labeling number $\lambda_d(G)$ of a graph G, which is the minimum cardinality k such that G has a k-$L(d, 1)$-labeling. In this paper, we determine the $L(d, 1)$-labeling number of the Cartesian product of a cycle and a path.

Keywords: k-$L(d, 1)$-labeling; Cartesian product; Path; Cycle; Power; Chromatic number

1. Introduction

The distance-two labeling problem of graphs proposed by Griggs and Roberts [26] is a variation of the frequency assignment problem introduced by Hale [16]. Suppose we are given a number of transmitters or stations. The $L(d, 1)$-labeling problem is to assign frequencies (nonnegative integers) to the transmitters so that "close" transmitters must receive different frequencies and "very close" transmitters must receive frequencies that are at least two frequencies apart.

An $L(d, 1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$ and $|f(u) - f(v)| \geq d$ if $d(u, v) = 1$. For a nonnegative integer k, a k-$L(2, 1)$-labeling is an $L(d, 1)$-labeling such that no label is greater than k. The $L(d, 1)$-labeling number of G, denoted by $\lambda_d(G)$, is the smallest number k such that G has a k-$L(d, 1)$-labeling. To simplify the notations, we often use $\lambda(G)$ to represent the $L(2, 1)$-labeling number of G.

The $L(d, 1)$-labeling problem has been studied for more than a decade. Griggs and Yeh [15] showed that the $L(2, 1)$-labeling problem is NP-complete for general graphs. They proved that $\lambda(G) \leq \Delta^2(G) + 2\Delta(G)$ and conjectured that $\lambda(G) \leq \Delta^2(G)$ for general graphs. Chang and Kuo [3] proved that $\lambda(G) \leq \Delta^2(G) + \Delta(G)$ and gave a linear-time algorithm for the $L(2, 1)$-labeling problem on cographs and a polynomial-time algorithm on trees. For further studies on the $L(2, 1)$-labelings, see [2,5–7,9,11–15,19–21,24,27–29,31]. (Variations, [17,22,23,25,30] for circular distance-two labelings and [1,4] for $L(d, 1)$-labelings on digraphs.)

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The purpose of this paper is to study the $L(d, 1)$-labeling problem for the Cartesian product of a cycle and a path. Given two graphs G and H, the Cartesian product of these two graphs, denoted by $G\Box H$, is defined by $V(G\Box H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$ and $E(G\Box H) = \{(u, x)(v, y) \mid (u = v, xy \in E(H))$ or $(uv \in E(G), x = y)\}$.

The $L(d, 1)$-labeling number of products of graphs was studied in [10,13,14,18–21,28,29]. The following are the known results on the λ-number of the products of two graphs.

Theorem 1 (*[29]*).

(a) $\lambda(P_m\Box P_n) = \begin{cases} 5, & \text{if } n = 2 \text{ and } m \geq 3; \\ 6, & \text{if } m, n \geq 3. \end{cases}$

(b) *If $n \geq 2$, $m_i \geq 3$ for all i, and $m_i \geq 4$ for at least two distinct i, then* $\lambda(P_{m_1}\Box P_{m_2}\Box \cdots \Box P_{m_n}) = 2n + 1$.

Theorem 2 (*[13,14]*). *If $n, m \geq 2$, then* $\lambda(K_m\Box K_n) = \begin{cases} 4, & \text{if } n = m = 2; \\ nm - 1, & \text{otherwise}. \end{cases}$

Theorem 3 (*[19–21]*). *If $n \geq 4$ and $m \geq 3$, then*

(a) $\lambda(C_m\Box P_2) = \begin{cases} 5, & \text{if } m \equiv 0 \pmod 3; \\ 6, & \text{if } m \not\equiv 0 \pmod 3. \end{cases}$

(b) $\lambda(C_m\Box P_3) = \begin{cases} 6, & \text{if } m \neq 4, 5; \\ 7, & \text{if } m = 4 \text{ or } 5. \end{cases}$

(c) $\lambda(C_m\Box P_n) = \begin{cases} 6, & \text{if } m \equiv 0 \pmod 7; \\ 7, & \text{if } m \not\equiv 0 \pmod 7. \end{cases}$

Theorem 4 (*[19,21,28]*). *If $m, n \geq 3$, then*

$$\lambda(C_m\Box C_n) = \begin{cases} 6, & \text{if } m, n \equiv 0 \pmod 7; \\ 7, & \text{if } \{n, m\} \in A; \\ 8, & \text{otherwise} \end{cases}$$

where $A = \{(3, i) \mid i \geq 3, i \text{ is odd or } i = 4, 10\} \cup \{(5, i) \mid i = 5, 6, 9, 10, 13, 17\} \cup \{(6, 7), (6, 11), (7, 9), (9, 10)\}$.

The following lemmas are useful.

Lemma 5 (*[15]*). *If G contains three vertices of degree k such that one of them is adjacent to the other two, then* $\lambda(G) \geq k + 2$.

Lemma 6 (*[21]*). *If f is a k-$L(d, 1)$-labeling of a graph G, then the function $f' : V(G) \longrightarrow \{0, 1, \ldots, k\}$ defined by $f'(v) = k - f(v)$ for all $v \in V(G)$ is also a k-$L(d, 1)$-labeling of G.*

Theorem 7 (*[8]*).

$$\lambda_d(C_m) = \begin{cases} 2d, & \text{if } m \text{ is odd}, d \geq 2; \\ d + 2, & \text{if } m \equiv 0 \pmod 4, d \geq 2; \\ d + 2, & \text{if } m \equiv 2 \pmod 4, d = 2; \\ d + 3, & \text{if } m \equiv 2 \pmod 4, d \geq 3. \end{cases}$$

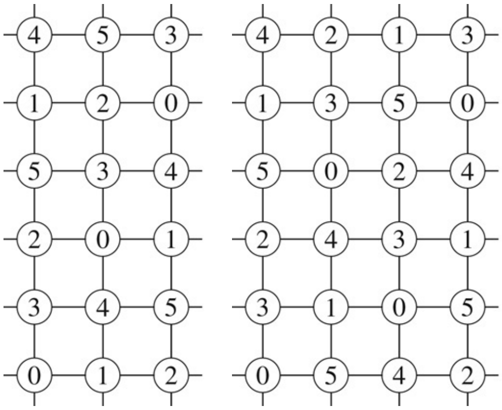
Lemma 8. *Suppose f is a $2d$-$L(d, 1)$-labeling of odd cycle C_{2k+1}, then $f(v) = d$ for some vertex v in C_{2k+1} and $f(N(v)) = \{0, 2d\}$.*

Proof. Suppose $f(v) \neq d$ for each vertex v in C_{2k+1} and let $C_{2k+1} : v_1, \ldots, v_{2k+1}$. W.L.O.G., we assume that $f(v_1) < d$. Then $f(v_{2i}) > d + 1$ and $f(v_{2i+1}) < d$. This implies $|f(v_1) - f(v_{2k+1})| \leq d - 1 < d$, a contradiction. So $f(v) = d$ for some vertex v in C_{2k+1}. It is trivial that $f(N(v)) = \{0, 2d\}$. ■

Lemma 9. *If f is a $(2d + 1)$-$L(d, 1)$-labeling of C_{2k+1}, then $f(v) = d$ or $d + 1$ for some vertex v in C_{2k+1}.*

Proof. The proof is similar to the proof of Lemma 8. ■

In this paper, we determine $\lambda_d(C_m\Box P_n)$ when $d = 1$ or $d \geq 3$. The vertices of $C_m\Box P_n$ are denoted by (i, j), where $i \in Z_m$ and $0 \leq j \leq n - 1$. We let $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $\deg(v) = |N(v)|$ for a vertex v in a graph G.



(a) $C_3\square P_6$. (b) $C_4\square P_6$.

Fig. 1. A 5-$L(1, 1)$-labeling of $C_m\square P_n$ for $m \not\equiv 0 \pmod 5$.

2. The $L(1, 1)$-labeling number of $C_m\square P_n$

In this section, we determine $\lambda_1(C_m\square P_n)$ for all $m \geq 3$ and $n \geq 2$. When $n = 2$, Georges and Mauro [12] give the following results.

$$\lambda_1(C_m\square P_2) = \begin{cases} 3, & \text{if } m \equiv 0 \pmod 4; \\ 5, & \text{if } m = 3, 6; \\ 4, & \text{otherwise.} \end{cases}$$

Theorem 10. *For all $n \geq 3$,*

$$\lambda_1(C_m\square P_n) = \begin{cases} 4, & \text{if } m \equiv 0 \pmod 5; \\ 5, & \text{if } m \not\equiv 0 \pmod 5. \end{cases}$$

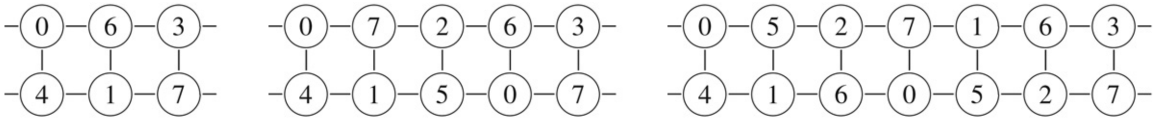
Proof. Since the degree of the vertex $(1, 1)$ is 4, we have $\lambda_1(C_m\square P_n) \geq 4$. The labeling f with $f(i, j) = [(i + 2j) \bmod 5]$ is a 4-$L(1, 1)$-labeling of $C_{5k}\square P_n$. So $\lambda_1(C_m\square P_n) = 4$ when $m \equiv 0 \pmod 5$. By combining Fig. 1(a) and 1(b), we have a 5-$L(1, 1)$-labeling of $C_m\square P_n$ when $m \not\equiv 0 \pmod 5$. Thus, $\lambda_1(C_m\square P_n) \leq 5$.

Let g be a 4-$L(1, 1)$-labeling of $C_m\square P_n$. Suppose $g(i, 1) = g(j, 1)$ for some $j - i \leq 4$. W.L.O.G., we may assume that $g(0, 1) = g(3, 1) = 0$ or $g(0, 1) = g(4, 1) = 0$, and let $g(0, 0) = 1$, $g(1, 1) = 2$, and $g(0, 2) = 3$. If $g(0, 1) = g(3, 1) = 0$, then $\{g(1, 0), g(2, 0)\} = \{3, 4\}$. This implies $g(2, 1) = 1$ and $g(1, 2) = 4$. Then there are no labels that can be assigned to $g(2, 2)$, a contradiction. So $g(0, 1) = g(4, 1) = 0$. Then we have $g(1, 0) \in \{3, 4\}$. If $g(1, 0) = 3$, then $\{g(2, 1), g(1, 2)\} = \{1, 4\}$. This implies $g(2, 2) = 0$, $g(2, 0) = 4$, $g(2, 1) = 1$, $g(1, 2) = 4$, $g(3, 1) = 3$, and $g(3, 0) = 2$. Then there are no labels that can be assigned to $g(3, 2)$, a contradiction. So $g(1, 0) = 4$. Then $g(1, 2) = 1$, $g(2, 1) = 3$, $g(2, 0) = 0$, $g(2, 2) = 4$, $g(3, 1) = 1$, and $g(3, 2) = 2$. But there are no labels that can be assigned to $g(3, 0)$, a contradiction. So $g(i, 1) \neq g(j, 1)$ when $|j - i| \leq 4$. Thus, $\lambda_1(C_m\square P_n) \geq 5$ if $m \not\equiv 0 \pmod 5$ and $m \geq 6$. Since $\lambda_1(C_m\square P_n) = 4$ implies $\lambda_1(C_{mk}\square P_n) \leq 4$ for any positive integer k. So $\lambda_1(C_m\square P_n) \geq 5$ if $m = 3, 4$. Thus, $\lambda_1(C_m\square P_n) = 5$ if $m \not\equiv 0 \pmod 5$. ■

The *chromatic number* $\chi(G)$ of G is the minimum number of colors of a proper coloring of G, where a *proper coloring* of G is a mapping from $V(G)$ to positive integer set such that adjacent vertices receive different values. Given a positive integer k, the k-power of a graph G is the graph G^k with $V(G^k) = V(G)$ and

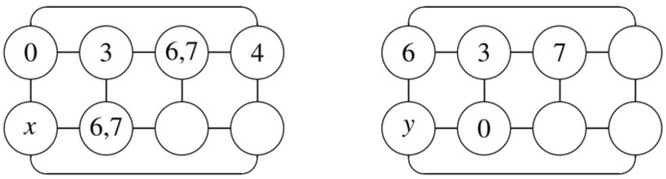
$$E(G^k) = \{uv : 1 \leq d(u, v) \leq k\}.$$

It is easy to see that [8] $\lambda_1(G) = \chi(G^2)$. Thus we have the following corollary.



(a) $C_3\Box P_2$. (b) $C_5\Box P_2$. (c) $C_7\Box P_2$.

Fig. 2. A 7-$L(3, 1)$-labeling of $C_m\Box P_2$ for $m \neq 4$.



(a) (b)

Fig. 3. A 7-$L(3, 1)$-labeling of $C_4\Box P_2$.

Corollary 11. *Let $n \geq 2$ and $G = C_m\Box P_n$. Then*

$$\chi(G^2) = \begin{cases} 3, & \text{if } n = 2 \text{ and } m \equiv 0 \pmod 4; \\ 5, & \text{if } n = 2 \text{ and } m = 3, 6; \\ 5, & \text{if } n \geq 3 \text{ and } m \not\equiv 0 \pmod 5; \\ 4, & \text{otherwise}. \end{cases}$$

We list $\lambda_1(C_m\Box P_n)$, $n \geq 2$, in Table 1.

3. The $L(3, 1)$-labeling number of $C_m\Box P_n$

In this section, we determine $\lambda_3(C_m\Box P_n)$ for all $m \geq 3$ and $n \geq 2$.

Lemma 12. $\lambda_d\ (C_m\Box P_2) \geq 2d + 1$ *if m is odd.*

Proof. Suppose f is a $2d$-$L(d, 1)$-labeling of $C_m\Box P_2$. By Lemma 8, we may assume that $f(0, 0) = 0$, $f(1, 0) = d$, and $f(2, 0) = 2d$ since m is odd. But the numbers, ranging from 0 to $2d$, cannot be assigned to $f(1, 1)$, a contradiction. So, $\lambda_d\ (C_m\Box P_2) \geq 2d + 1$. ■

Theorem 13. $\lambda_3\ (C_m\Box P_2) = 7$ *if $m \neq 4$.*

Proof. By Fig. 2, we have $\lambda_d\ (C_m\Box P_2) \leq 7$ when $m \neq 4$. Suppose f is a 6-$L(3, 1)$-labeling of $C_m\Box P_2$. Since the degree of each vertex is 3, the number 4, 3 and 2 cannot be assigned in f. W.L.O.G., we may assume that $f(1, 0) = 6$. Then the labels of $(0, 0)$, $(2, 0)$ or $(1, 1)$ must be either 0 or 1. Since they should be different, it is impossible. So $\lambda_3\ (C_m\Box P_2) = 7$ if $m \neq 4$. ■

Theorem 14. $\lambda_3\ (C_4\Box P_2) = 8$.

Proof. Let f be a 7-$L(3, 1)$-labeling of $C_4\Box P_2$. Suppose $f(v) = 3$ for some vertex v. W.L.O.G., we have such labeling as Fig. 3.

The number from 0 to 7 cannot be assigned to either x or y, a contradiction. Thus $f(v) \neq 3$ for all v in $C_m\Box P_2$. By Lemma 6, $f(v) \neq 4$ for all v in $C_m\Box P_2$. W.L.O.G., we suppose that $0 \leq f(0, 0) \leq 2$. Then $f(1, 0)$, $f(3, 0)$ and $f(0, 1)$ must be 5, 6, or 7. Thus the labels of $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(3, 1)$ must be chosen from 0, 1, or 2. In addition, since the labels assigned to $(0, 0)$, $(1, 1)$, $(2, 0)$, and $(3, 1)$ should be all different, it is impossible. So $\lambda_3\ (C_4\Box P_2) \geq 8$.

Fig. 4 shows an 8-$L(3, 1)$-labeling of $C_4\Box P_2$. So $\lambda_3\ (C_4\Box P_2) = 8$. ■

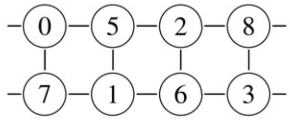


Fig. 4. An 8-$L(3,1)$-labeling of $C_4 \square P_2$.

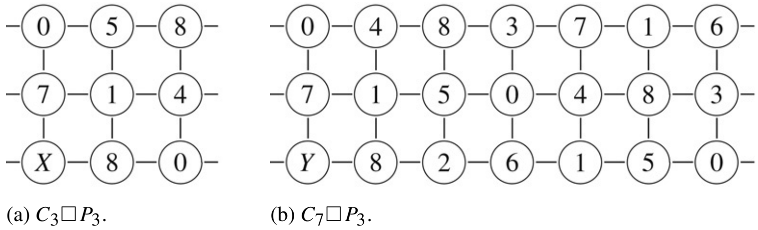


Fig. 5. An 8-$L(3,1)$-labeling of $C_m \square P_3$ for $m \neq 4, 5, 8, 11$.

Lemma 15. $\lambda_3\,(C_m \square P_3) \geq 8$.

Proof. Suppose f is a 7-$L(3,1)$-labeling of $C_m \square P_3$. Then $f(i,1) \neq 2, 3, 4, 5$ for all i. We may assume that $f(0,1) = 0$ and $f(2,1) = 1$. Then $f(\{(1,1),(3,1)\}) = \{6,7\}$. If $f(1,0) = 3$ or 4, then no labels can be assigned to label $(2,0)$, a contradiction. The cases are similar for $f(1,2) = 3$ or 4. So $\lambda_3\,(C_m \square P_3) \geq 8$. ■

Lemma 16.

$$\lambda_3\,(C_m \square P_3) = \begin{cases} 8, & \textit{if } m \neq 4, 5, 8, 11; \\ 9, & \textit{if } m = 4, 5, 8, 11. \end{cases}$$

Proof. According to the combination of Fig. 5(a) and (b) by choosing suitable X and Y, we have $\lambda_3\,(C_m \square P_3) \leq 8$ for $m \neq 4, 5, 8, 11$. Note that $X, Y = 3$ or 4. So $\lambda_3\,(C_m \square P_3) = 8$ when $m \neq 4, 5, 8, 11$. With to the aid of computer program implementation, we have $\lambda_3\,(C_m \square P_3) = 9$ where $m = 4, 5, 8, 11$. Fig. 6 shows a 9-$L(3,1)$-labeling of $C_m \square P_3$ when $m = 4, 5, 8, 11$. ■

Theorem 17. $\lambda_3(C_{3k} \square P_n) = 8$ *if* $n \geq 4$ *and* $k \geq 1$.

Proof. By Lemma 15, we have $\lambda_3(C_{3k} \square P_n) \geq 8$. Fig. 7 shows an 8-$L(3,1)$-labeling f of $C_{3k} \square P_3$. Define the labeling g in $C_{3k} \square P_n$ by $g(i,j) = f(i,j)$ if $j \leq 2$ and $g(i,j) = g(i-1, j-3)$ if $j > 2$. Then g is an 8-$L(3,1)$-labeling of $C_{3k} \square P_n$. Thus $\lambda_3(C_{3k} \square P_n) = 8$ when $n \geq 4$. ■

In the rest of this section, we will show that $\lambda_3(C_m \square P_n) = 9$ when $n \geq 4$ and $m \not\equiv 0 \pmod 3$. We suppose that $n \geq 4$ and f is an 8-$L(3,1)$-labeling of $C_m \square P_n$. We let T_i denote the 4-cycle induced by vertices $(i,1), (i,2), (i+1,1)$, and $(i+1,2)$ for $i \in Z_m$; and let $f(T_i) = \{f(i,1), f(i,2), f(i+1,1), f(i+1,2)\}$. Let $v = (i,j)$ be a vertex in $C_m \square P_n$ and $a, b \in N(v)$. We denote $a \perp b$ if in $C_m \square P_n$, there is a 4-cycle containing a, b, v. Otherwise, we denote $a // b$.

Lemma 18. *Let* $i \in Z_m$ *and* $0 < j < n-1$.

(1) *If* $f(i,j) = 3$, *then* $\{f(a), f(b)\} \neq \{0,6\}, \{7,8\}$ *when* $a \perp b$, *and* $\{f(a), f(b)\} = \{0,6\}, \{7,8\}$ *when* $a // b$.
(2) *If* $f(i,j) = 4$, *then* $\{f(a), f(b)\} \neq \{0,8\}, \{1,7\}$ *when* $a \perp b$, *and* $\{f(a), f(b)\} = \{0,8\}, \{1,7\}$ *when* $a // b$.
(3) *If* $f(i,j) = 5$, *then* $\{f(a), f(b)\} \neq \{0,1\}, \{2,8\}$ *when* $a \perp b$, *and* $\{f(a), f(b)\} = \{0,1\}, \{2,8\}$ *when* $a // b$.

Proof. We only show the case (1), while the cases (2) and (3) are similar.

W.L.O.G., we assume that $a = (i+1, j), b = (i, j+1)$ when $a \perp b$. Suppose $f(a) = 0$ and $f(b) = 6$. Then no label can be assigned to $(i+1, j+1)$. Thus, $\{f(a), f(b)\} \neq \{0,6\}$. If $\{f(a), f(b)\} = \{7,8\}$, then

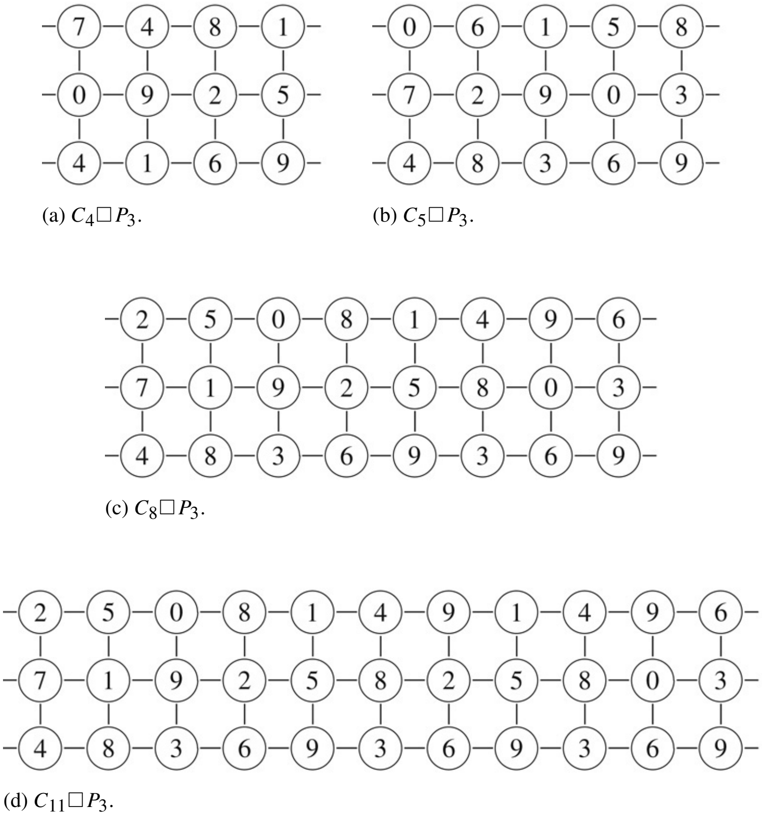


Fig. 6. A 9-$L(3, 1)$-labeling of $C_m\Box P_3$ for $m = 4, 5, 8, 11$.

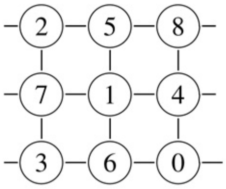


Fig. 7. A 9-$L(3, 1)$-labeling of $C_{3k}\Box P_3$.

$\{f(i-1, j), f(i, j-1)\} = \{0, 6\}$. It is impossible. So $\{f(a), f(b)\} \neq \{7, 8\}$. It is trivial that $f(N(i, j)) = \{0, 6, 7, 8\}$. Therefore $\{f(a), f(b)\} = \{0, 6\}$ or $\{7, 8\}$ when $a//b$. ■

It is easy to check that $f(T_i) = \{0, 1, 4, 5\}, \{0, 1, 4, 6\}, \{0, 1, 4, 7\}, \{0, 1, 4, 8\}, \{0, 1, 5, 6\}, \{0, 1, 5, 7\}, \{0, 1, 5, 8\}, \{0, 1, 6, 7\}, \{0, 1, 6, 8\}, \{0, 1, 7, 8\}, \{0, 2, 5, 6\}, \{0, 2, 5, 7\}, \{0, 2, 5, 8\}, \{0, 2, 6, 7\}, \{0, 2, 6, 8\}, \{0, 2, 7, 8\}, \{0, 3, 4, 7\}, \{0, 3, 4, 8\}, \{0, 3, 5, 8\}, \{0, 3, 6, 7\}, \{0, 3, 6, 8\}, \{0, 3, 7, 8\}, \{0, 4, 5, 8\}, \{0, 4, 7, 8\}, \{1, 2, 5, 6\}, \{1, 2, 5, 7\}, \{1, 2, 5, 8\}, \{1, 2, 6, 7\}, \{1, 2, 6, 8\}, \{1, 2, 7, 8\}, \{1, 3, 6, 7\}, \{1, 3, 6, 8\}, \{1, 3, 7, 8\}, \{1, 4, 5, 8\}, \{1, 4, 7, 8\}, \{2, 3, 6, 7\}, \{2, 3, 6, 8\}, \{2, 3, 7, 8\}, \{2, 4, 7, 8\}$, or $\{3, 4, 7, 8\}$.

In fact, only 9 sets are possible among them. We show them in the following lemma.

Lemma 19. *For each* i, $f(T_i) = \{0, 1, 4, 6\}, \{0, 2, 5, 6\}, \{0, 3, 4, 7\}, \{0, 3, 5, 8\}, \{1, 2, 5, 7\}, \{1, 3, 6, 7\}, \{1, 4, 5, 8\}, \{2, 3, 6, 8\}$, *or* $\{2, 4, 7, 8\}$.

Proof. This lemma will be proved by the following cases.

1. $f(T_i) \neq \{0, 3, 4, 8\}, \{0, 4, 5, 8\}$.
2. $f(T_i) \neq \{0, 1, 4, 5\}, \{0, 1, 5, 6\}, \{0, 1, 5, 7\}, \{0, 1, 5, 8\}, \{3, 4, 7, 8\}, \{2, 3, 7, 8\}, \{1, 3, 7, 8\}, \{0, 3, 7, 8\}$.
3. $f(T_i) \neq \{0, 1, 6, 7\}, \{1, 2, 7, 8\}$.
4. $f(T_i) \neq \{0, 1, 6, 8\}, \{0, 2, 7, 8\}$.
5. $f(T_i) \neq \{0, 1, 4, 7\}, \{1, 4, 7, 8\}$.
6. $f(T_i) \neq \{0, 1, 4, 8\}, \{0, 4, 7, 8\}$.
7. $f(T_i) \neq \{0, 1, 7, 8\}$.
8. $f(T_i) \neq \{0, 2, 5, 7\}, \{1, 3, 6, 7\}$.
9. $f(T_i) \neq \{0, 2, 5, 8\}, \{0, 3, 6, 8\}$.
10. $f(T_i) \neq \{0, 3, 6, 7\}, \{1, 2, 5, 8\}$.
11. $f(T_i) \neq \{0, 2, 6, 7\}, \{1, 2, 6, 8\}$.
12. $f(T_i) \neq \{0, 2, 6, 8\}$.
13. $f(T_i) \neq \{1, 2, 5, 6\}, \{2, 3, 6, 7\}$.
14. $f(T_i) \neq \{1, 2, 6, 7\}$.

Case 1. $f(T_i) \neq \{0, 3, 4, 8\}, \{0, 4, 5, 8\}$.

W.L.O.G., we may assume that $f(i, 1) = 4$. Then $f(i + 1, 2) = 3$. So $\{f(i, 2), f(i + 1, 1)\} = \{0, 6\}$. It is a contradiction to Lemma 18(2). Thus $f(T_i) \neq \{0, 3, 4, 8\}$. By Lemma 6, we have $f(T_i) \neq \{0, 4, 5, 8\}$.

Case 2. $f(T_i) \neq \{0, 1, 4, 5\}, \{0, 1, 5, 6\}, \{0, 1, 5, 7\}, \{0, 1, 5, 8\}, \{3, 4, 7, 8\}, \{2, 3, 7, 8\}, \{1, 3, 7, 8\}, \{0, 3, 7, 8\}$.

The proof here is similar to that for Case 1 by Lemma 18(3).

Case 3. $f(T_i) \neq \{0, 1, 6, 7\}, \{1, 2, 7, 8\}$.

W.L.O.G., we may assume that $f(i, 1) = 0$ and $f(i + 1, 2) = 1$. If $f(i, 2) = 6$, then $f(i + 1, 1) = 7$ and $f(i - 1, 2) = 2$ or 3. Assume that $f(i - 1, 2) = 2$. Then $f(i, 3) = 3$, $f(i + 1, 3) = 8$. Since $f(i + 2, 1) = 2, 3$, or 4, we have $f(i + 2, 2) = 5$ and $f(i + 2, 1) = 2$. So $f(i + 2, 3) = 0$, a contradiction to Lemma 18(3). By Lemma 18, the arguments of other cases are all similar. Thus, $f(T_i) \neq \{0, 1, 6, 7\}$. By Lemma 6, we have $f(T_i) \neq \{1, 2, 7, 8\}$.

Case 4. $f(T_i) \neq \{0, 1, 6, 8\}, \{0, 2, 7, 8\}$.

The proof is similar to that for Case 3.

Case 5. $f(T_i) \neq \{0, 1, 4, 7\}, \{1, 4, 7, 8\}$.

W.L.O.G., we may assume that $f(i, 1) = 0$ and $f(i + 1, 2) = 1$. If $f(i, 2) = 4$, then $f(i + 1, 1) = 7$. By Lemma 18(2), we have $f(i - 1, 2) = 7$. Then $f(i - 1, 1) = 3$. By Lemma 18(1), $f(i - 1, 0) = 8$. So $f(i, 0) = 5$, $f(i + 1, 0) = 2$, $f(i + 2, 1) = 3$ or 4, $f(i + 2, 0) = 6$ or 8, $f(i + 2, 2) = 6$ or 8. By Lemma 18(1) and (2), it is impossible.

If $f(i, 2) = 7$, then $f(i + 1, 1) = 4$. By Lemma 18(2), we have $f(i + 1, 0) = 7$. Then $f(i, 0) = 3$, $f(i - 1, 1) = 5, 6$, or 8 and $f(i-1, 0) = 6$ or 8. So, $f(i-1, 1) = 5$, $f(i-1, 0) = 8$. By Lemma 18(3), $f(i-1, 2) = 2$, $f(i-2, 1) = 1$. Then $f(i - 2, 0) = 4$, $f(i - 1, 3) = 6$ or 8, $f(i - 2, 2) = 6$ or 8, $f(i - 3, 1) = 6, 7$, or 8, and $f(i - 3, 2) = 3$ or 4. If $f(i - 3, 2) = 4$, then $f(i - 2, 2) = 8$, $f(i - 1, 3) = 6$, $f(i - 3, 1) = 7$. By Lemma 18(2), $f(i - 3, 3) = 1$. Then no labels can be assigned to $(i - 2, 3)$, a contradiction. For the case of $f(i - 3, 2) = 3$, we have $f(i - 2, 2) = 6$ or 8 and $f(i - 1, 3) = 6$ or 8. Then $f(i - 2, 3) = 0$ and $f(i - 3, 3) = 6, 7$, or 8. By Lemma 18 (1), we have $f(i-2, 2) = 6$, $f(i-1, 3) = 8$, $f(i-3, 3) = 7$, $f(i-3, 1) = 8$, $f(i-4, 2) = 0$, $f(i-4, 1) = 5$, and $f(i-3, 0) = 0$. Then no labels can be assigned to $f(i - 4, 0)$. So $f(T_i) \neq \{0, 1, 4, 7\}$. By Lemma 6, $f(T_i) \neq \{1, 4, 7, 8\}$.

Case 6. $f(T_i) \neq \{0, 1, 4, 8\}, \{0, 4, 7, 8\}$.

W.L.O.G.,we may assume that $f(i, 1) = 0$ and $f(i + 1, 2) = 1$. If $f(i, 2) = 4$, then $f(i + 1, 1) = 8$. By Lemma 18(2), we have $f(i - 1, 2) = 7$. Then $f(i - 1, 1) = 3$. By Lemma 18(1), $f(i - 1, 0) = 8$. So $f(i, 0) = 5$, $f(i + 1, 0) = 2$, $f(i + 2, 1) = 3, 4$, or 5, $f(i + 2, 0) = 6$ or 7, $f(i + 2, 2) = 6$ or 7. By Lemma 18, it is impossible. Assume that $f(i, 2) = 8$. Then $f(i + 1, 1) = 4$. By Lemma 18(2), we have $f(i + 1, 0) = 7$ and $f(i + 2, 1) = 8$. Then $f(i + 2, 2) = 5$, $f(i, 0) = 3$, $f(i - 1, 0) = 6$ or 8, $f(i - 1, 1) = 5, 6$, or 7. Then $f(i - 1, 0) = 8$, $f(i - 1, 1) = 5$ and $f(i - 1, 2) = 2$. By Lemma 18 and some similar arguments, we have for all $k \geq 1$, $f(i + 1 - 3k, 2) = 6$, $f(i - 3k, 2) = 3$, $f(i - 3k - 1, 2) = 0$, $f(i + 1 - 3k, 1) = 1$, $f(i - 3k, 1) = 7$, $f(i - 3k - 1, 1) = 4$. Since m is finite, f cannot form an 8-$L(3, 1)$-labeling of $C_m \square P_n$, a contradiction. So $f(T_i) \neq \{0, 1, 4, 8\}$. By Lemma 6, $f(T_i) \neq \{0, 4, 7, 8\}$.

Case 7. $f(T_i) \neq \{0, 1, 7, 8\}$.

Table 1
$\lambda_1\,(C_m \square P_n)$, $n \geq 2$

n	m	$\lambda_1\,(C_m \square P_n)$
$n = 2$	$m \equiv 0 \pmod 4$	3
$n = 2$	$m \not\equiv 0 \pmod 4$ and $m \neq 3, 6$	4
$n = 2$	$m \equiv 3, 6$	5
$n \geq 3$	$m \equiv 0 \pmod 5$	4
$n \geq 3$	$m \not\equiv 0 \pmod 5$	5

Table 2
$\lambda_3\,(C_m \square P_n)$

n	m	$\lambda_3(C_m \square P_n)$
2	$m \neq 4$	7
2	$m = 4$	8
3	$m \neq 4, 5, 8, 11$	8
3	$m = 4, 5, 8, 11$	9
≥ 4	$m \equiv 0 \pmod 3$	8
≥ 4	$m \not\equiv 0 \pmod 3$	9

Table 3
$\lambda_d\,(C_m \square P_n)$, $d \geq 4$

n	m	$\lambda_d(C_m \square P_n)$, $d \geq 4$
2	m is odd	$2d + 1$
2	$m \equiv 0 \pmod 6$	$d + 4$
2	m is even and $m \not\equiv 0 \pmod 6$	$d + 5$
3	m is odd	$2d + 2$
3	m is even	$d + 6$
$n \geq 4$	m is odd	$2d + 2$
$n \geq 4$	$m \equiv 0 \pmod 4$	$d + 6$
$n \geq 4$	$m \equiv 2 \pmod 4$	$d + 6$, when $d = 4$
$4 \leq n \leq 2k + 2$	$m = 4k + 2$	$d + 6$, when $d \geq 5$
$n \geq 2k + 3$	$m = 4k + 2$	$d + 7$, when $d \geq 5$

Assume that $f(i, 1) = 0$ and $f(i + 1, 2) = 1$. If $f(i, 2) = 7$ and $f(i + 1, 1) = 8$, then $f(i + 2, 1) = 2, 3, 4$, or 5 and $f(i + 2, 2) = 4, 5$, or 6. So $f(i + 2, 1) = 2$ or 3 and $f(i + 2, 2) = 5$ or 6. Similarly, we have $f(i, 0) = 5$ or 6, $f(i + 1, 0) = 2$ or 3, $f(i - 1, 1) = 5$ or 6, $f(i - 1, 2) = 2$ or 3, $f(i, 3) = 2$ or 3, and $f(i + 1, 3) = 5$ or 6. By Lemma 18, we have $f(i + 2, 1) = 2$, $f(i - 1, 1) = 6$. Then $f(i, 0) = 5$ and $f(i + 1, 0) = 3$, a contradiction. The proof for the case of $f(i, 2) = 8$ is similar. Thus $f(T_i) \neq \{0, 1, 7, 8\}$.

Case 8. $f(T_i) \neq \{0, 2, 5, 7\}, \{1, 3, 6, 7\}$.

Assume that $f(i, 1) = 0$ and $f(i + 1, 2) = 2$. If $f(i, 2) = 5$, then $f(i + 1, 1) = 7$. By Lemma 18(3), we have $f(i - 1, 2) = 8$ and $f(i, 3) = 1$. Then $f(i - 1, 3) = 4$, $f(i - 1, 1) = 3$. By Lemma 18(1), we have $f(i - 1, 0) = 7$, $f(i, 0) = 4$, $f(i + 1, 0) = 1$, $f(i + 2, 1) = 3$ or 4, and $f(i + 2, 2) = 6$ or 8. By Lemma 18(1) and (2), $f(i + 2, 1) \neq 3, 4$, a contradiction. The other cases are similar. So $f(T_i) \neq \{0, 2, 5, 7\}$. By Lemma 6, $f(T_i) \neq \{1, 3, 6, 7\}$.

Case 9. $f(T_i) \neq \{0, 2, 5, 8\}, \{0, 3, 6, 8\}$.

Assume that $f(i, 1) = 0$, $f(i + 1, 2) = 2$. If $f(i, 2) = 5$, then $f(i + 1, 1) = 8$. By Lemma 18(3), we have $f(i - 1, 2) = 8$ and $f(i, 3) = 1$. Then $f(i - 1, 3) = 4$ and $f(i - 1, 1) = 3$. By Lemma 18(1), $f(i - 1, 0) = 7$. Therefore $f(i, 0) = 4$, $f(i + 1, 0) = 1$ and $f(i + 2, 2) = 3, 4$, or 5. By Lemma 18, we have $f(i + 2, 2) \neq 3, 4, 5$, a contradiction. For the case of $f(i, 2) = 8$, we have $f(i + 1, 1) = 5$. By Lemma 18(3), we have $f(i + 2, 1) = 1$ and $f(i + 1, 0) = 8$. Then $f(i + 2, 0) = 4$, $f(i, 0) = 3$, $f(i - 1, 0) = 6$ or 7, and $f(i - 1, 1) = 4, 6$, or 7. This implies $f(i - 1, 1) = 4$ and $f(i - 1, 2) = 1$. Thus $f(T_i) = \{0, 1, 4, 8\}$, a contradiction to Case 6. So $f(T_i) \neq \{0, 2, 5, 8\}$. By Lemma 6, $f(T_i) \neq \{0, 3, 6, 8\}$.

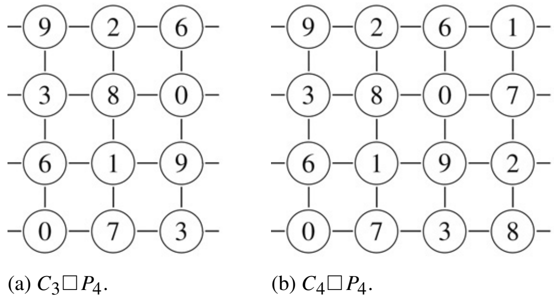


Fig. 8. A 9-$L(3, 1)$-labeling of $C_m\Box P_n$ for $n \geq 4$ and $m \neq 5$.

Case 10. $f(T_i) \neq \{0, 3, 6, 7\}, \{1, 2, 5, 8\}$.

Assume that $f(i, 1) = 0$, $f(i + 1, 2) = 3$. If $f(i, 2) = 7$, then $f(i + 1, 1) = 6$. By Lemma 18(1), we have $f(i + 1, 3) = 0$. Then $f(i, 3) = 4$, $f(i - 1, 2) = 1$ or 2, $f(i - 1, 3) = 8$, and $f(i - 1, 1) = 4$ or 5. This implies $f(T_{i-1}) = \{0, 1, 4, 7\}, \{0, 1, 5, 7\}$, or $\{0, 2, 5, 7\}$. By Cases 2, 5, and 8, it is impossible. So $f(i, 2) \neq 7$. Then $f(i, 2) = 6$ and $f(i + 1, 1) = 7$. By Lemma 18(1), we have $f(i + 2, 2) = 0$ and $f(i + 1, 3) = 8$. Then $f(i + 2, 1) = 4$ and $f(i + 2, 3) = 5$. By Lemma 18(2), we have $f(i + 2, 0) = 8$ and $f(i + 3, 1) = 1$. By Cases 5 and 6, $f(i + 3, 2) = 6$. By similar argument, we have $f(i + 3k + 1, 1) = 7$, $f(i + 3k + 2, 1) = 4$, and $f(i + 3k + 3, 1) = 1$ for all $k = 0, 1, 2, \ldots$. It is a contradiction to $f(i, 1) = 0$. So $f(T_i) \neq \{0, 3, 6, 7\}$. By Lemma 6, we have $f(T_i) \neq \{1, 2, 5, 8\}$.

Case 11. $f(T_i) \neq \{0, 2, 6, 7\}, \{1, 2, 6, 8\}$.

Assume that $f(i, 1) = 0$ and $f(i - 1, 2) = 2$. If $f(i, 2) = 6$ and $f(i - 1, 1) = 7$, $f(i - 1, 2) = 1$ or 3, and $f(i - 1, 1) = 4, 5$, or 8. So $f(T_i) = \{0, 1, 4, 6\}$ or $\{0, 1, 5, 6\}, \{0, 1, 6, 8\}$, or $\{0, 3, 6, 8\}$. By Cases 2, 4, and 9, we have $f(T_i) = \{0, 1, 4, 6\}$, that is, $f(i - 1, 1) = 4$ and $f(i - 1, 2) = 1$. By Lemma 18(2), we have $f(i - 1, 0) = 7$. Then $f(i, 0) = 3$. Now no number can be assigned to $(i + 1, 0)$, a contradiction. The other cases are similar. So $f(T_i) \neq \{0, 2, 6, 7\}$. By Lemma 6, $f(T_i) \neq \{1, 2, 6, 8\}$.

Case 12. $f(T_i) \neq \{0, 2, 6, 8\}$.

The proof is similar to that for Case 11.

Case 13 $f(T_i) \neq \{1, 2, 5, 6\}, \{2, 3, 6, 7\}$.

The proof is similar to that for Case 11.

Case 14. $f(T_i) \neq \{1, 2, 6, 7\}$. The proof is similar to that for Case 11. ■

Theorem 20. $\lambda_3(C_m\Box P_n) = 9$ *if* $n \geq 4$ *and* $m \not\equiv 0 \pmod 3$.

Proof. Let f be a 8-$L(3, 1)$-labeling of $C_m\Box P_n$. For each possible set in Lemma 19, we have $f(T_i) \neq f(T_{i+1})$ for each i and it implies a recurrence of size $3k$. Thus, $m \equiv 0 \pmod 3$. So, $\lambda_3(C_m\Box P_n) \geq 9$ if $m \not\equiv 0 \pmod 3$ and $n \geq 4$.

From Fig. 8(a) and (b), we have a 9-$L(3, 1)$-labeling of $C_m\Box P_n$ if $n \geq 4$ and $m \neq 5$. Fig. 9 shows a 9-$L(3, 1)$-labeling of $C_5\Box P_n$. So, $\lambda_3(C_m\Box P_n) = 9$ if $n \geq 4$ and $m \not\equiv 0 \pmod 3$. ■

We list $\lambda_3(C_m\Box P_n)$, $n \geq 2$; in Table 2.

4. The $L(d, 1)$-labeling number of $C_m\Box P_n$, $d \geq 4$

In this section, we determine $\lambda_d(C_m\Box P_n)$ for all $n \geq 2$ and $d \geq 4$.

Theorem 21. $\lambda_d\ (C_m\Box P_2) = 2d + 1$ *if* m *is odd and* $d \geq 4$.

Proof. By Lemma 12, $\lambda_d\ (C_m\Box P_2) \geq 2d + 1$ if m is odd. So, by Fig. 10, we have $\lambda_d\ (C_m\Box P_2) = 2d + 1$ if m is odd and $d \geq 4$. ■

Theorem 22. $\lambda_d\ (C_m\Box P_2) = d + 4$ *if* $d \geq 4$ *and* $m \equiv 0 \pmod 6$.

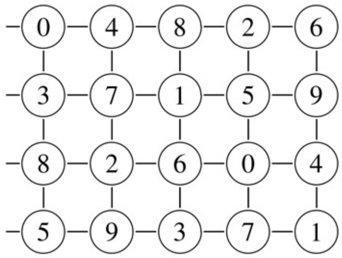
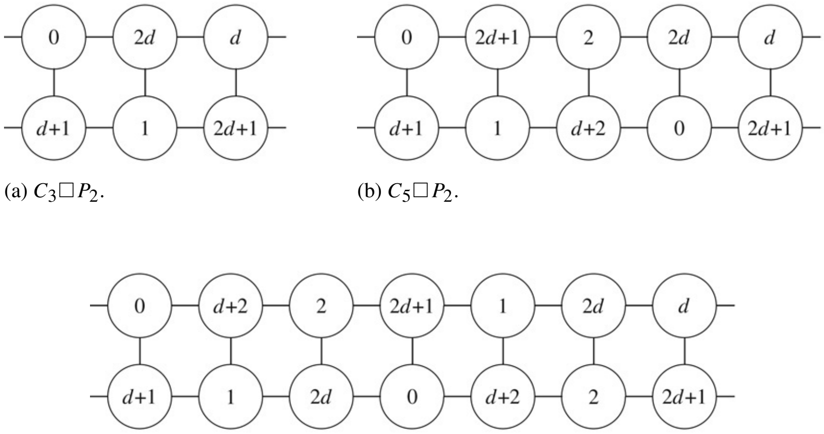


Fig. 9. A 9-$L(3, 1)$-labeling of $C_5 \Box P_n$.



(a) $C_3 \Box P_2$.

(b) $C_5 \Box P_2$.

(c) $C_7 \Box P_2$.

Fig. 10. A $(2d + 1)$-$L(d, 1)$-labeling of $C_m \Box P_2$ for $d \geq 4$.

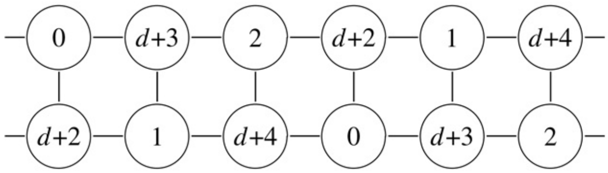


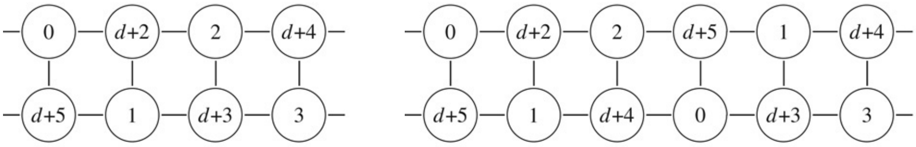
Fig. 11. A $(d + 4)$-$L(d, 1)$-labeling of $C_{6k} \Box P_2$ for $d \geq 4$.

Proof. Suppose f is a $(d+3)$-$L(d, 1)$-labeling of $C_m \Box P_2$. We may assume that $f(v) = d+3$ for some v in $C_m \Box P_2$. Let $f(N(v)) = \{a, b, c\}$. Since the degree of each vertex is 3, the number 3 and 2 cannot be assigned in f. Therefore a, b, and c could be either 0 or 1 only. It contradicts the fact that a, b, and c should be 3 distinct numbers. So, $\lambda_d\,(C_m \Box P_2) \geq d + 4$ when $d \geq 4$.

Fig. 11 shows a $(d + 4)$-$L(d, 1)$-labeling of $C_{6k} \Box P_2$ for $d \geq 4$. So, $\lambda_d\,(C_m \Box P_2) = d + 4$ if $d \geq 4$ and $m \equiv 0 \pmod 6$. ■

Theorem 23. *$\lambda_d\,(C_m \Box P_2) = d + 5$ if $d \geq 4$, m is even, and $m \not\equiv 0 \pmod 6$.*

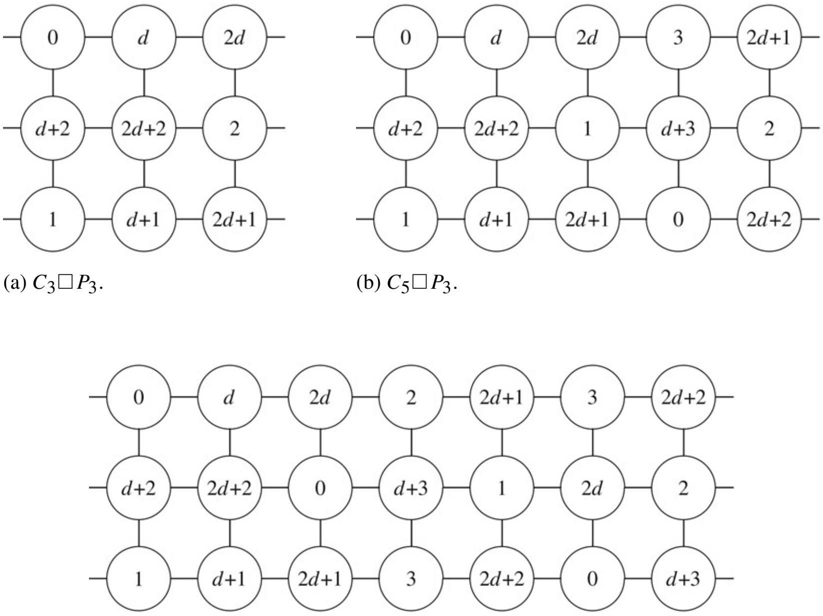
Proof. Suppose f is a $(d+4)$-$L(d, 1)$-labeling of $C_m \Box P_2$. It is easy to check that $3, 4, \ldots, d+1$ cannot be assigned in f. Let $a_i = f(i, j)$ if $i + j$ is even and $b_i = f(i, j)$ if $i + j$ is odd. W.L.O.G., we may assume that $a_0 = 0$. Then all of a_i could be 0, 1, or 2 only. Since a_{i-1}, a_i, and a_{i+1} must be different, we have $m \equiv 0 \pmod 6$. So $\lambda_d\,(C_m \Box P_2) \geq d + 5$ if $d \geq 4$ and $m \not\equiv 0 \pmod 6$.



(a) $C_4\Box P_2$.

(b) $C_6\Box P_2$.

Fig. 12. A $(d+5)$-$L(d,1)$-labeling of $C_m\Box P_2$ for $d \geq 4$ and $m \equiv 0 \pmod 2$.



(a) $C_3\Box P_3$.

(b) $C_5\Box P_3$.

(c) $C_7\Box P_3$.

Fig. 13. A $(2d+2)$-$L(d,1)$-labeling of $C_m\Box P_3$ for $d \geq 4$ and $m \neq 4$.

By combining two labelings in Fig. 12, we have $\lambda_d(C_m\Box P_2) \leq d+5$ if m is even and $d \geq 4$. Thus $\lambda_d(C_m\Box P_2) = d+5$ if $d \geq 4$, m is even, and $m \not\equiv 0 \pmod 6$. ■

Theorem 24. $\lambda_d(C_m\Box P_3) = 2d+2$ *if* m *is odd and* $d \geq 4$.

Proof. Suppose that f is a $(2d+1)$-$L(d,1)$-labeling of $C_m\Box P_3$. By Lemma 6 and Lemma 9, we may assume that $f(1,1) = d$. Then $f(\{(0,1),(1,0),(2,1),(1,2)\}) = \{0, 2d, 2d+1\}$. It is impossible since f is a $L(d,1)$-labeling. So $\lambda_d(C_m\Box P_3) \geq 2d+2$. By combining Fig. 13(a) with either 13(b) or 13(c), we have $\lambda_d(C_m\Box P_3) \leq 2d+2$ when $d \geq 4$ and $m \neq 4$. So $\lambda_d(C_m\Box P_3) = 2d+2$ if m is odd and $d \geq 4$. ■

Theorem 25. $\lambda_d(C_m\Box P_3) = d+6$ *if* m *is even and* $d \geq 4$.

Proof. Suppose $d \geq 5$ and f is a $(d+5)$-$L(d,1)$-labeling of $C_m\Box P_3$. In Fig. 14, the degree of X is 4, we have $f(X) \in \{0,1,2,d+3,d+4,d+5\}$. W.L.O.G., suppose that $f(X) \in \{0,1,2\}$. It is easy to see that $f(v) \in \{0,1,2,3,d+2,d+3,d+4,d+5\}$ for all $v \in C_m\Box P_3$. So, we have $f(a_i) \in \{d+2,d+3,d+4,d+5\}$. Because $a_2, a_4 \neq d+2$, we may let $a_1 = d+2$. Since $f(b_i) \leq 3$ for $i = 1,2,3$, and $f(b_2)$, $f(b_3)$, $f(x) \leq 2$, we have $f(b_1) = 3$. Similarly, we have $f(c_1) = d+2$. But $f(a_1) = d+2$, it is impossible. Thus $\lambda_d(C_m\Box P_3) \geq d+6$ if $d \geq 5$.

Suppose g is a 9-$L(4,1)$-labeling of $C_m\Box P_3$. It is not hard to check that $g(i,1) \in \{0,1,2,7,8,9\}$ for all i. We may let $g(0,1), g(2,1), g(4,1) \in \{7,8,9\}$ and $g(1,1), g(3,1), g(5,1) \in \{0,1,2\}$. Then $g(1,2) \in \{4,5,6\}$ or

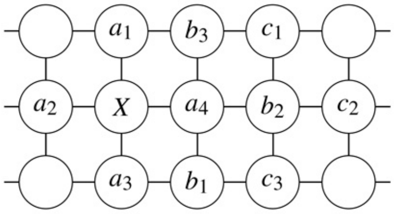


Fig. 14. $C_m \square P_3$.

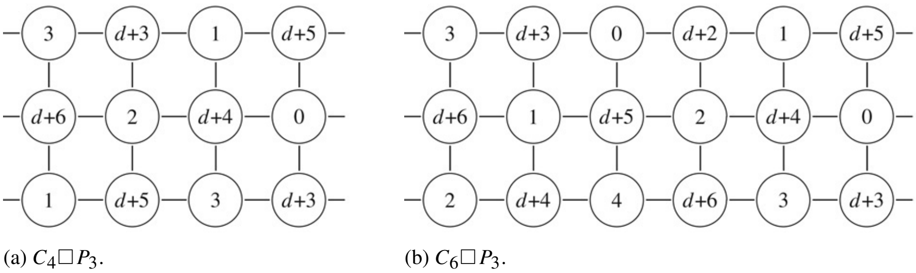


Fig. 15. A $(d+5)$-$L(d, 1)$-labeling of $C_m \square P_3$ for $d \geq 4$ and $m \equiv 0 \pmod 2$.

$g(1, 0) \in \{4, 5, 6\}$. W.L.O.G., we may assume that $g(1, 2) \in \{4, 5, 6\}$. Then $g(2, 2) \in \{0, 1, 2\}$ and $g(2, 0) \in \{3, 4, 5\}$. This implies that $g(3, 2) \in \{4, 5, 6\}$ and $g(0, 2), g(4, 2) \in \{0, 1, 2\}$. This implies that $g(1, 2) = g(3, 2) = 6$, a contradiction. Thus $\lambda_4(C_m \square P_3) \geq 10 = d + 6$.

By combining Fig. 15(a) and (b), we have a $(d+6)$-$L(d, 1)$-labeling of $C_m \square P_3$ if m is even and $d \geq 4$. So $\lambda_d(C_m \square P_3) = d + 6$ if m is even and $d \geq 4$. ■

Theorem 26. *$\lambda_d(C_m \square P_n) = 2d + 2$ if m is odd, $d \geq 4$, and $n \geq 4$.*

Proof. By Theorem 24, we have $\lambda_d(C_m \square P_n) \geq 2d+2$ if m is odd, $d \geq 4$, and $n \geq 4$. Let f be the $(2d+2)$-$L(d, 1)$-labeling of $C_m \square P_3$ in Theorem 24. We define a labeling g in $C_m \square P_n$ such that $g(i, j) = f(i, j)$ if $j \leq 2$ and $g(i, j) = g(i-1, j-3)$ if $j > 2$, where $i-1$ means $((i-1) \bmod m)$. Then g is a $(2d+2)$-$L(d, 1)$-labeling of $C_m \square P_n$. So $\lambda_d(C_m \square P_n) = 2d + 2$ when m is odd, $d \geq 4$, and $n \geq 4$. ■

Theorem 27. *$\lambda_d(C_m \square P_n) = d + 6$ if $m \equiv 0 \pmod 4$ and $n, d \geq 4$.*

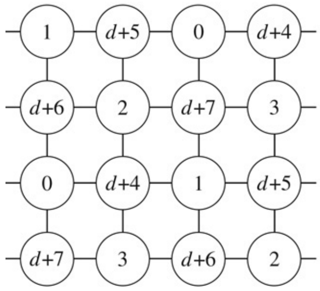
Proof. By Theorem 25, we have $\lambda_d(C_m \square P_n) \geq d + 6$ if $d \geq 4$. Let f be the $(d+6)$-$L(d, 1)$-labeling of $C_m \square P_3$, which is shown in Theorem 25. We define a labeling g in $C_m \square P_n$ such that

$$g(i, j) = \begin{cases} f(i, j), & \text{if } j \leq 1; \\ g(i+2, j-2), & \text{if } j > 1. \end{cases}$$

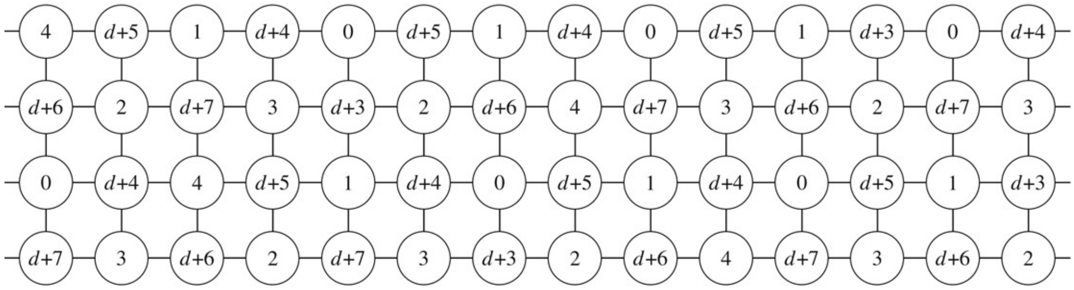
Then g is a $(d+6)$-$L(d, 1)$-labeling of $C_m \square P_n$. So $\lambda_d(C_m \square P_n) = d + 6$ if $n, d \geq 4$ and $m \equiv 0 \pmod 4$. ■

Lemma 28. *Suppose $n, d \geq 4$ and $m \equiv 2 \pmod 4$. Then $\lambda_d(C_m \square P_n) \leq d + 7$.*

Proof. By combining Fig. 16(a) and (b), we have a $(d+7)$-$L(d, 1)$-labeling of $C_m \square P_n$ if $m = 14 + 4k, k = 0, 1, 2, \ldots$. By Fig. 17, we have $\lambda_d(C_m \square P_n) \leq d + 7$ if $m = 6$. For the case of $m = 10$, we define a labeling f such



(a) $C_4\Box P_4$.



(b) $C_{14}\Box P_4$.

Fig. 16. A $(d+7)$-$L(d,1)$-labeling of $C_m\Box P_n$ for $m = 4k+14$, $k \geq 0$.

that

$$f(i,j) = \begin{cases} d+3+\left(\left(\mathrm{i}+\dfrac{j}{2}\right) \bmod 5\right), & \text{if } i,\ j \text{ even;} \\ d+3+\left(\left(\mathrm{i}+2+\dfrac{j+1}{2}\right) \bmod 5\right), & \text{if } i,\ j \text{ odd;} \\ \left(\mathrm{i}+2+\dfrac{j-1}{2}\right) \bmod 5, & \text{if } i \text{ even, } j \text{ odd;} \\ \left(\mathrm{i}-1+\dfrac{j}{2}\right) \bmod 5, & \text{if } i \text{ odd, } j \text{ even.} \end{cases}$$

Then f is a $(d+7)$-$L(d,1)$-labeling of $C_{10}\Box P_n$. So $\lambda_d(C_m\Box P_n) \leq d+7$ when $m \equiv 2 (\bmod\ 4)$. ■

Theorem 29. *Suppose $d \geq 5$, $n \geq 4$, and $m = 4k+2$ for some positive integer k. Then*

$$\lambda_d(C_m\Box P_n) = \begin{cases} d+6, & \text{if } 4 \leq n \leq 2k+2; \\ d+7, & \text{if } n \geq 2k+3. \end{cases}$$

Proof. Define a labeling f in $C_{4k+2}\Box P_{2k+2}$ such that:

(1) $f(0,0) = f(2k+1, 2k+1) = 4$.

(2) If $0 \leq i \leq k$, $0 \leq j \leq k-i$, and $(i,j) \neq (0,k)$, then $f(2i+4j+2, 2i) = 0$.

(3) If $0 \leq i \leq k$ and $0 \leq j < k-i$, then $f(2i+4j+3, 2i+1) = 1$, $f(2i+4j+4, 2i) = 2$, and $f(2i+4j+5, 2i+1) = 3$.

(4) If $0 \leq i \leq k$, $k-i \leq j \leq k$, and $(i,j) \neq (k,k)$, then $f(2i+4j+3, 2i+1) = 2$.

(5) If $0 \leq i \leq k$ and $k-i \leq j < k$, then $f(2i+4j+5, 2i+1) = 0$, $f(2i+4j+4, 2i) = 1$, and $f(2i+4j+6, 2i) = 3$.

(6) If $p+q$ is odd, then $f(p,q) = d+6-f(p-3,q)$.

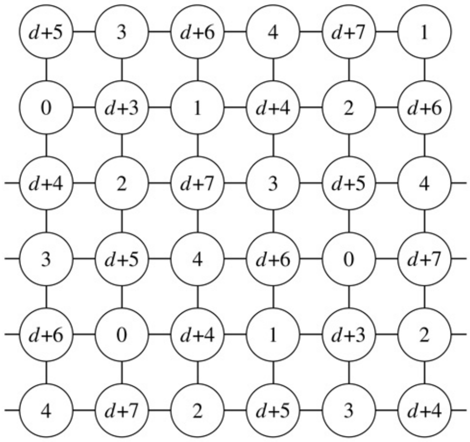


Fig. 17. A $(d+7)$-$L(d,1)$-labeling of $C_6\square P_n$.

We can check that f is a $(d+6)$-$L(d,1)$-labeling of $C_{4k+2}\square P_{2k+2}$. So $\lambda_d(C_{4k+2}\square P_n)\le d+6$ if $n\le 2k+2$. By Theorem 25, $\lambda_d(C_{4k+2}\square P_n)=d+6$ when $4\le n\le 2k+2$ and $d\ge 5$.

Suppose f is a $(d+6)$-$L(d,1)$-labeling of $C_{4k+2}\square P_{2k+3}$. The degree of (i,j) is 4 if $j=1,2,\dots,2k+1$. So $f(i,j)\in\{0,1,2,3,d+3,d+4,d+5,d+6\}$. W.L.O.G., we can suppose that $f(i,j)\in\{0,1,2,3\}$ if $1\le j\le 2k+1$ and $i+j$ is odd. Let $a,b,c,h\in\{0,1,2,3\}$, where a,b,c,h are different. Suppose $f(2k,1)=a$, $f(2k-1,2)=b$, $f(2k,3)=c$, $f(2k+1,2)=h$. Since f is a $(d+6)$-$L(d,1)$-labeling of $C_{4k+2}\square P_{2k+3}$, $f(j,2)\ne b,h$ for some j. W.L.O.G., we suppose that $j=2k+3$. Then $f(2k+3,2)=a$ or c. Assume that $f(2k+3,2)=a$. Then

$$\begin{cases} f(2k-2i,1+2i)=a, & 0\le i\le k-1;\\ f(2k-2i-1,2+2i)=b, & 0\le i\le k-1;\\ f(2k-2i+2,1+2i)=c, & 0\le i\le k;\\ f(2k+1-2i,2+2i)=h, & 0\le i\le k-1;\\ f(2k-2i+4,1+2i)=b, & 1\le i\le k;\\ f(2k+3-2i,2+2i)=a, & 0\le i\le k-1. \end{cases}$$

So we have

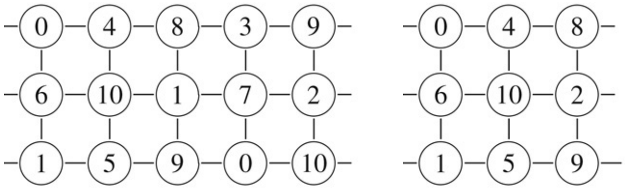
$$\begin{cases} f(2k+3+4j-i,i)=c \text{ or } h, & 2j+1\le i\le 2k+1, 1\le j\le k;\\ f(2k+5+4j-i,i)=a \text{ or } b, & 2j+2\le i\le 2k+1, 1\le j<k;\\ f(2k+3+4j-i,i)=a \text{ or } b, & 1\le i\le 2j, 1\le j\le k;\\ f(2k+5+4j-i,i)=c \text{ or } h, & 1\le i\le 2j+1, 0\le j<k. \end{cases}$$

Since $f(1,2k)=b$ and $f(2,2k+1)=c$, we have $f(4k+1,2k)=a$, $f(0,2k+1)=h$, and $f(4k+1,2k-2)=b$. This implies $f(0,2k-1)=c$ and $f(4k,2k-1)=h$. Since $f(4k+1,2k-2)=b$ and $f(4k,2k-1)=h$, we have $f(4k-1,2k-2)=a$ and $f(4k,2k-3)=c$. Thus,

$$\begin{cases} f(4k+1-2j,2k-2j)=a, & 1\le j\le k-2;\\ f(4k+3-2j,2k-2j)=b, & 1\le j\le k-1;\\ f(4k+2-2j,2k-2j+1)=h, & 1\le j\le k-1;\\ f(4k-2-2j,2k-2j-1)=c, & 1\le j\le k-2. \end{cases}$$

Then no label can be assigned to $f(2k+4,1)$, a contradiction. For the case of $f(2k+3,2)=c$, the proof is similar. So, we have $\lambda_d(C_{4k+2}\square P_n)\ge d+7$. By Lemma 28, $\lambda_d(C_{4k+2}\square P_n)=d+7$ if $n\ge 2k+3$ and $d\ge 5$. ■

Theorem 30. $\lambda_4(C_m\square P_n)=10$ *if* $m\equiv 2 \pmod 4$ *and* $n\ge 4$.



(a) $C_5 \square P_3$. (b) $C_3 \square P_3$.

Fig. 18. A 10-$L(4, 1)$-labeling of $C_m \square P_3$ for $m \geq 3$ and $m \neq 4, 7$.

Proof. By Theorem 25, we have $\lambda_4(C_m \square P_n) \geq 10$. By combining Fig. 18 gives a 10-$L(4, 1)$-labeling f of $C_m \square P_3$ for $m \geq 3$ and $m \neq 4, 7$. Define a labeling g in $C_m \square P_n$ such that $g(i, j) = f(i, j)$ if $j \leq 2$ and $g(i, j) = g(i+1, j-3)$ if $j > 2$. Then g is a 10-$L(4, 1)$-labeling of $C_m \square P_n$. So, $\lambda_4(C_m \square P_n) = 10$ when $m \equiv 2 (\bmod 4)$ and $n \geq 4$. ■

We list $\lambda_d(C_m \square P_n)$, $d \geq 4$, in Table 3.

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